# Perfect Explicit Model-Following Control Solution to Imperfect Model-Following Control Problems

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**For cases in which perfect model-following is not possible for a particular desired model, a class of candidate models is defined that can be followed perfectly by the given plant. A candidate model that most closely matches the dynamics of the desired model is then determined through constrained parameter optimization. The result is perfect model-following of a model that has an eigenstructure which resembles that of the desired model. In the development of this method, a new variation on perfect model-following control law development is shown. This method explicitly displays the feed-forward and feedback gains that determine the system error dynamics, which may be artibrarily selected by conventional pole placement methods if the plant is completely controllable. The method is applied to a problem involving the linearized lateral-directional equations of motion of the B-26 airplane. The results show that a candidate model can be found that has virtually the same dynamic behavior as the desired model, and that it can be followed perfectly by the original plant with arbitrarily assigned error dynamics.**

## Nomenclature

$A$ = system matrix, $n \times n$
$B$ = control input matrix, $n \times m$
$e$ = error vector, $n \times 1$
$G$ = matrix relating rows of a matrix to a basis for its row space
$K_e$ = error dynamics gain matrix, $m \times n$
$M$ = modal matrix
$r$ = yaw rate
$T$ = transformation matrix
$u$ = control vector, $m \times 1$
$W$ = a weighting matrix
$x$ = state vector, $n \times 1$
$\beta$ = sideslip angle
$\Delta$ = matrix of differences between $A$ matrices and $B$ matrices
$\delta_a$ = aileron input
$\delta_r$ = rudder input
$\phi$ = bank angle
$\dot{\phi}$ = roll rate

*Superscipts*

$T$ = matrix transpose
$w$ = left generalized inverse
$-1$ = matrix inverse
$+$ = pseudo inverse
$*$ = conjugate operation (matrix conjugate transpose)

*Subscripts*

$c$ = candidate model
cm = control model
$m$ = desired model
$p$ = plant

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## Introduction

PREVIOUS analyses of the model-following problem have developed criteria for determining when perfect following is possible.[1] The solution to the perfect model-following control problem is straightforward, and gains for dynamic matching may be obtained directly from the linearized equations of motion.[2] Trajectory following may be achieved by error-correcting dynamics that are assignable by the designer.[3] However, models that can be perfectly followed by a given plant are the exception. When perfect following is not possible, linear quadratic optimization techniques are frequently employed.[4-9]

This paper presents an alternative approach to the imperfect model-following problem. It is based on the idea that a model can be found from a class of candidate models that not only satisfies the requirements for perfect following but also retains, in some sense, the characteristics of the desired model. The class of candidate models will be defined based on criteria for perfect following. The design objective is defined as the selection of a candidate model whose eigenstructure is near (in a weighted least squares sense) to that of the desired model. The selection of the candidate model is based on a parameter optimization problem that is solved subject to constraints that ensure perfect model-following.

The procedure is illustrated through application to a variation of an often analyzed problem regarding the linearized lateral-directional equations of motion of the B-26 airplane.

## Problem Statement

Given the linearized equations of motion of the plant

$$\dot{x}_p = A_p x_p + B_p u_p \tag{1}$$

and of the desired model

$$\dot{x}_m = A_m x_m + B_m u \tag{2}$$

finding the following is required: a characterization of all candidate models that can be followed perfectly by the plant; a candidate model whose characteristics closely match those of the desired model; and the control law that implements the following of the candidate model for all control inputs and that corrects for errors due to differences in initial conditions or disturbances in the respective trajectories of the plant and the candidate model.

## Perfect Model-Following

### Standard Form

At this point, it is convenient to introduce a *standard form* for the plant and model. Consider a plant and model whose linearized equations of motion are given by

$$\dot{x}_p = \begin{bmatrix} \dot{x}_p^1 \\ \dot{x}_p^2 \end{bmatrix} = \begin{bmatrix} A^1 \\ A_p^2 \end{bmatrix} \begin{bmatrix} x_p^1 \\ x_p^2 \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} u_p$$

$$x_p^1 : (n-m) \times 1, \qquad x_p^2 : m \times 1$$

$$A^1 : (n-m) \times n, \qquad A_p^2 : m \times n \tag{3}$$

$$\dot{x}_m = \begin{bmatrix} \dot{x}_m^1 \\ \dot{x}_m^2 \end{bmatrix} = \begin{bmatrix} A^1 \\ A_m^2 \end{bmatrix} \begin{bmatrix} x_m^1 \\ x_m^2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_m^2 \end{bmatrix} u$$

$$x_m^1 : (n-m) \times 1, \qquad x_m^2 : m \times 1$$

$$A^1 : (n-m) \times n, \qquad A_m^2 : m \times n, \qquad B_m^2 : m \times m \tag{4}$$

The following characteristics of these equations are emphasized:

1) The plant and the model are of the same order ($n$).

2) For a plant with $m$ controls, there are exactly $m$ plant equations that depend on these controls, and these controls appear through the identity matrix for those $m$ equations.

3) For the $n-m$ plant equations that do not directly depend on the controls, there are $n-m$ model equations in which the controls do not appear. These $n-m$ equations are identical with the corresponding plant equations (the submatrix $A^1$ is the same for both plant and model).

4) The remaining $m$ equations are arbitrary, including $B_m^2$. The externally applied control vector $u$ is of any order less than or equal to $m$.

Such a system satisfies the usual criteria for perfect model-following given by Erzberger et al.[1] Additionally, all systems that satisfy the criteria for perfect model-following may be put in the form of Eqs. (3) and (4) by simultaneous similarity transformations on the plant and model equations. These assertions are proved as follows.

For the case of full-state feedback, criteria for perfect model-following may be written as

$$[I - B_p B_p^+] B_m = [0] \tag{5}$$

$$[I - B_p B_p^+][A_m - A_p] = [0] \tag{6}$$

where $B_m$, $A_m$, and $A_p$ are, in general, *not* in the form given by Eqs. (3) and (4). The superscript + indicates the pseudoinverse of the matrix. [Notation and definitions for generalized inverses follow Boullion and Odell.[10] $B^w$ is the left generalized inverse, whereas $B^+$ is the (unique) pseudoinverse. $B^w$ satisfies $(BB^wB) = B$, $(B^wBB^w) = B^w$, and $(B^wB)^* = (B^wB)$. $B^+$ additionally satisfies $(BB^+)^* = (BB^+)$. The use of the Penrose left pseudoinverse, which appears frequently in analyses of this sort, is not necessarily the inverse of choice.] In the form given by Eq. (3)

$$B_p^+ = B_p^T = [0 \ \vdots \ I]$$

$$[I - B_p B_p^+] = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

The proof that Eqs. (3) and (4) imply Eqs. (5) and (6) is immediate because the submatrix $A^1$ is the same for plant and model. To prove that Eqs. (5) and (6) imply the existence of a similarity transformation that results in Eqs. (3) and (4), we first consider the class $L$ of all left generalized inverses of $B_p$

$$L = \{B_p^w \mid [B_p B_p^w B_p] = B_p, \quad [B_p^w B_p B_p^w] = B_p^w, \quad [B_p^w B_p]^T = [B_p^w B_p]\} \tag{7}$$

and note that $B_p^+ \in L$. We partition $B_p$ as

$$B_p = \begin{bmatrix} B_p^1 \\ B_p^2 \end{bmatrix} \tag{8}$$

and require, without loss of generality (since $B_p$ is assumed to be of full rank), that $\mathrm{rank}(B_p^2) = m$. This implies that $\exists G \ni B_p^1 = G B_p^2$ or

$$B_p = \begin{bmatrix} G B_p^2 \\ B_p^2 \end{bmatrix} \tag{9}$$

We take a general form of the left inverse as[11]

$$B_p^w = [P \ \vdots \ B_p^{2^{-1}} - PG] \tag{10}$$

where $P$ is arbitrary. Using this form

$$[I - B_p B_p^w] = \begin{bmatrix} I - GB_p^2 P & \vdots & -[I - GB_p^2 P]G \\ -B_p^2 P & \vdots & B_p^2 PG \end{bmatrix} \tag{11}$$

We also need to partition the other matrices to conform with left multiplication by Eq. (11)

$$B_m = \begin{bmatrix} B_m^1 \\ B_m^2 \end{bmatrix}, \qquad A_m = \begin{bmatrix} A_m^1 \\ A_m^2 \end{bmatrix}, \qquad A_p = \begin{bmatrix} A_p^1 \\ A_p^2 \end{bmatrix} \tag{12}$$

Using Eq. (11) and performing the operations indicated by Eqs. (5) and (6), it is found that the conditions for perfect model-following hold if and only if

$$B_p^1 - G B_p^2 = [0] \tag{13a}$$

$$[A_m^1 - A_p^1] - G[A_m^2 - A_p^2] = [0] \tag{13b}$$

For convenience, introduce the matrix $\Delta$ such that

$$\Delta = \begin{bmatrix} \Delta^1 \\ \Delta^2 \end{bmatrix} \equiv \begin{bmatrix} A_m^1 - A_p^1 & \vdots & B_m^1 - B_p^1 \\ A_m^2 - A_p^2 & \vdots & B_m^2 - B_p^2 \end{bmatrix} \tag{14}$$

The conditions for perfect model-following are that

$$\Delta^1 = G \Delta^2 \tag{15}$$

The rest of the proof is constructive. Consider the similarity transformation represented by

$$T \equiv \begin{bmatrix} T^{11} & \vdots & T^{12} \\ T^{21} & \vdots & T^{22} \end{bmatrix} = \begin{bmatrix} T^{11} & \vdots & -T^{11} G \\ P & \vdots & B_p^{2^{-1}} - PG \end{bmatrix} \tag{16}$$

where $T^{11}$ is arbitrary, except that $T$ must be nonsingular. Now note that if a plant and model satisfy Eqs. (5) and (6), then Eq. (15) must be satisfied, and that Eq. (16) is the required similarity transformation.

Equations (3) and (4) will be referred to as the *standard form* for perfect model-following.

### Control Law

In the standard form, the plant equations of motion may be solved directly for the control:

$$u_p = \dot{x}_p^2 - A_p^2 x_p \tag{17}$$

With no error, perfect dynamic matching is realized by substituting the model state rates for the corresponding plant state rates in the control equation

$$u_p = \dot{x}_m^2 - A_p^2 x_p \tag{18}$$

To show that this results in perfect dynamic matching, consider the error between model and plant,

$$e = \begin{bmatrix} e^1 \\ e^2 \end{bmatrix} \equiv \begin{bmatrix} x_m^1 - x_p^1 \\ x_m^2 - x_p^2 \end{bmatrix} \tag{19}$$

and the error rate,

$$\dot{e} = \begin{bmatrix} \dot{x}_m^1 - \dot{x}_p^1 \\ \dot{x}_m^2 - \dot{x}_p^2 \end{bmatrix} = \begin{bmatrix} A^1 x_m - A^1 x_p \\ \dot{x}_m^2 - [A_p^2 x_p + u_p] \end{bmatrix}$$

$$= \begin{bmatrix} A^1[x_m - x_p] \\ \dot{x}_m^2 - [A_p^2 x_p + \dot{x}_m^2 - A_p^2 x_p] \end{bmatrix} = \begin{bmatrix} A^1 \\ 0 \end{bmatrix} e \tag{20}$$

The error dynamics may be assigned arbitrarily by modifying the model equations. This modified model is referred to as the control model, denoted by the subscript $cm$, and is defined as

$$\dot{x}_{\text{cm}} = \begin{bmatrix} \dot{x}_{\text{cm}}^1 \\ \dot{x}_{\text{cm}}^2 \end{bmatrix} \equiv \begin{bmatrix} \dot{x}_m^1 \\ \dot{x}_m^2 \end{bmatrix} + \begin{bmatrix} 0 \\ K_e \end{bmatrix} e \tag{21}$$

The control model is still in the standard form and is identical to the original model when the error is zero. The equation for the control law is

$$u_p = \dot{x}_{\text{cm}}^2 - A_p^2 x_p$$

$$= (\dot{x}_m^2 + K_e e) - A_p^2 x_p \tag{22}$$

The corresponding error dynamics become

$$\dot{e} = \begin{bmatrix} A^1 \\ -K_e \end{bmatrix} e \tag{23}$$

Equivalently,

$$\dot{e} = \begin{bmatrix} A^{11} & A^{12} \\ 0 & 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ I \end{bmatrix} u_e \tag{24}$$

where $A^{11}$ and $A^{12}$ are appropriately partitioned submatrices of $A^1$, and

$$u_e = -K_e e \tag{25}$$

The poles may be placed arbitrarily if

$$\text{rank} \begin{bmatrix} 0 & \vdots & A^{12} & \vdots & A^{11}A^{12} & \vdots & \cdots & \vdots & A^{11^{n-2}}A^{12} \\ I & \vdots & 0 & \vdots & 0 & \vdots & \cdots & \vdots & 0 \end{bmatrix} = n \tag{26}$$

$A_p$ has been partitioned as

$$A_p = \begin{bmatrix} A^{11} & A^{12} \\ A_p^{21} & A_p^{22} \end{bmatrix} \tag{27}$$

The plant is completely controllable if

$$\text{rank} \begin{bmatrix} 0 & \vdots & A^{12} & \vdots & A^{11}A^{12} & \vdots & \cdots & \vdots & A^{11^{n-2}}A^{12} \\ I & \vdots & A_p^{22} & \vdots & A_p^{22^2} & \vdots & \cdots & \vdots & A_p^{22^{n-1}} \end{bmatrix} = n \tag{28}$$

It follows that the errors are completely controllable if and only if the plant is. All of the plants considered in this paper are assumed to be completely controllable.

## Candidate Model Definition

It is now assumed that the desired model cannot be followed perfectly by the given plant. Consider as alternatives all models that can be perfectly followed by the given plant. These candidates must satisfy Eq. (15). With $\Delta_c \equiv [A_c - A_p \ \vdots \ B_c - B_p]$, $\Delta_c = [\Delta_c^1 \quad \Delta_c^2]^T$ (subscript $c$ is introduced to denote a candidate model), the requirement is that

$$[\Delta_c^1 - G\Delta_c^2] = [0] \tag{29}$$

Equation (29) represents $(n-m)\cdot(n+m)$ algebraic equations for the $n\cdot(n+m)$ undetermined parameters of the candidate system matrices $A_c$ and $B_c$. We wish to select from all candidate models the one that best approximates the dynamics of the desired model.

The dynamic responses of two systems may be compared through their respective modal decompositions. Associated with the desired model and the candidate model are their modal matrices $M_m$ and $M_c$, whose columns are the eigenvectors or generalized eigenvectors of the respective system.

With the transformation

$$x = Mq \tag{30}$$

the system dynamics are given by

$$\dot{q} = M^{-1}AMq + M^{-1}Bu$$

$$= \Lambda q + M^{-1}Bu \tag{31}$$

where $\Lambda$ is (for distinct eigenvalues) the diagonal matrix of system eigenvalues. If we now select the coefficients of $A_c$ and $B_c$ so that Eqs. (30) and (31) for the candidate model are as nearly as possible in some sense like those for the desired model, and such that their modal matrices are similar as well, then the dynamic characteristics of the candidate will be the best match possible. This problem can be cast in the form of a parameter optimization problem, where the cost to be minimized is

$$C = \|\Lambda_m - \Lambda_c\|_{W_1} + \|M_m - M_c\|_{W_2} + \|M_m^{-1}B_m - M_c^{-1}B_c\|_{W_3} \tag{32}$$

$W_1$, $W_2$, and $W_3$ are weighting matrices. They are included so that particular system eigenvalues, modes, or control response characteristics may be reproduced more faithfully at the expense of others (for example, fast modes at the expense of slow ones).

The cost given by Eq. (32) is to be minimized by selecting the elements of $A_c$ and $B_c$, subject to the constraints given by Eq. (29).

## Example

This example is based on the linearized lateral equations of motion of the B-26 airplane. All system matrices, including those of the desired model, are taken from Erzberger[1] and Tyler.[8] The desired model control matrix has been changed from that which is stated in the cited works, where it was equal to the plant control matrix. It is easily shown that the systems do not satisfy criteria for perfect model-following.

**Systems**

The states and controls of the system are $x = [\phi \quad \dot{\phi} \quad \beta \quad r]^T$, $u = [\delta_r \quad \delta_a]^T$. The $A$ and $B$ matrices are as follows:

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2.93 & -4.75 & -0.78 \\ 0.086 & 0 & -0.11 & -1.0 \\ 0 & -0.042 & 2.59 & -0.39 \end{bmatrix} \tag{33}$$

$$B_p = \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix} \tag{34}$$

$$A_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.0 & -73.14 & 3.18 \\ 0.086 & 0 & -0.11 & -1.0 \\ 0.0086 & 0.086 & 8.95 & -0.49 \end{bmatrix} \tag{35}$$

$$B_m = \begin{bmatrix} 0 & 0 \\ 0 & -3.91 \\ 0.175 & 0 \\ -2.53 & 0.31 \end{bmatrix} \tag{36}$$

The single change in the desired model control matrix [Eq. (36)] introduces nonzero entries in the right-hand sides of Eqs. (5) and (13a), ensuring that the desired model fails all the tests for perfect model-following.

**Candidate Models**

We take

$$B_p^2 = \begin{bmatrix} 0.035 & 0 \\ -2.53 & 0.31 \end{bmatrix}$$

For this selection,

$$G = \begin{bmatrix} 0 & 0 \\ -911.7 & -12.61 \end{bmatrix}$$

From Eq. (29), candidate models are therefore those for which

$$a_{c1j} - a_{p1j} = 0$$

$$(a_{c2j} - a_{p2j}) - C_1(a_{c3j} - a_{p3j}) - C_2(a_{c4j} - a_{p4j}) = 0 \tag{37}$$

$$b_{c1j} - b_{p1j} = 0$$

$$(b_{c2j} - b_{p2j}) - C_1(b_{c3j} - b_{p3j}) - C_2(b_{c4j} - b_{p4j}) = 0 \tag{38}$$

$$j = 1 \cdots 4, \qquad C_1 = -911.7, \qquad C_2 = -12.61$$

**Solution**

The problem was cast as a parameter optimization problem as described earlier. It was solved using a general nonlinear programming problem solver with finite difference gradients. The variables were the 18 undetermined elements of the candidate system and control matrices, and the eight equality constraints were as defined by Eqs. (37) and (38). The line searches used by the optimization procedure were sensitive to the cost associated with differences in the modal control matrices and required that this portion of the cost be deweighted considerably. This yielded the following candidate system:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.215 & -0.677 & -73.91 & 7.142 \\ 0.086 & -0.0037 & -0.123 & -1.003 \\ 0.0203 & 0.0486 & 8.989 & -0.801 \end{bmatrix} \tag{39}$$

$$B_c = \begin{bmatrix} 0 & 0 \\ 0.0405 & -3.914 \\ 0.0346 & 0 \\ -2.501 & 0.310 \end{bmatrix} \tag{40}$$

Table 1 presents the eigenvalues, eigenvectors, and modal control matrices for this solution and for those of the desired model.

The candidate model has eigenvalues, eigenvectors, and modal control matrices that are virtually the same as those of the desired model. The most notable differences occur in the least significant components of the eigenvectors and should have minimal effect on the similarity of the dynamic response of the desired and candidate models.

**Table 1 Eigenstructure comparison**

| | Candidate model | | Desired model | |
|---|---|---|---|---|
| Eigenvalues | $-1.023$ | | $-1.023$ | |
| | $-0.2887-j2.942$ | | $-0.2882-j2.942$ | |
| | $-0.2887+j2.942$ | | $-0.2882+j2.942$ | |
| | $-0.2729\times10^{-3}$ | | $-0.2752\times10^{-3}$ | |
| | Magnitude | Angle, deg | Magnitude | Angle, deg |
| First eigenvector | $-0.9772$ | 0.0 | $-0.9772$ | 0.0 |
| | 1.0 | 0.0 | 1.0 | 0.0 |
| | $-0.1017\times10^{-2}$ | 0.0 | $-0.3473\times10^{-2}$ | 0.0 |
| | $-0.8837\times10^{-1}$ | 0.0 | $-0.8721\times10^{-1}$ | 0.0 |
| Second eigenvector | 0.3383 | 95.6 | 0.3383 | 95.6 |
| | 1.0 | 0.0 | 1.0 | 0.0 |
| | $0.4103\times10^{-1}$ | 114.0 | $0.4262\times10^{-1}$ | 110.7 |
| | 0.1201 | $-173.4$ | 0.1231 | $-176.1$ |
| Fourth eigenvector | 1.0 | 0.0 | 1.0 | 0.0 |
| | $-0.2729\times10^{-3}$ | 0.0 | $-0.2752\times10^{-3}$ | 0.0 |
| | $0.5317\times10^{-2}$ | 0.0 | $0.3725\times10^{-2}$ | 0.0 |
| | $0.8505\times10^{-1}$ | 0.0 | $0.8559\times10^{-1}$ | 0.0 |

$$M_c^{-1}B_c$$

$$\begin{bmatrix} -18.07 & -1.642 \\ 9.053-j5.729 & -1.136+j0.429 \\ 9.053+j5.729 & -1.136-j0.429 \\ -20.92 & -1.391 \end{bmatrix}$$

$$M_m^{-1}B_m$$

$$\begin{bmatrix} -18.04 & -1.645 \\ 9.018-j5.796 & -1.133+j0.435 \\ 9.018+j5.796 & -1.133-j0.435 \\ -20.94 & -1.389 \end{bmatrix}$$

**Control Law Formulation**

The transformation to standard form was taken as [cf Eq. (16)]

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.0968\times10^{-3} & 1 & 1.3834\times10^{-2} \\ 0 & -3.134\times10^{-2} & 0 & -0.3952 \\ 1 & -0.2558 & 0 & 0 \end{bmatrix} \tag{41}$$

The states now are $y_p$ and $y_c$, where $y = Tx$. In standard form, we have

$$TA_pT^{-1} = \begin{bmatrix} 3.909 & 0 & 0 & -3.909 \\ 0.383 & -7.938\times10^{-2} & 2.543 & -0.297 \\ 0.369 & -0.875 & -0.482 & -0.369 \\ 6.777 & 1.215 & -0.462 & -6.777 \end{bmatrix} \tag{42a}$$

$$TB_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{42b}$$

**Simulations**

The systems were simulated to determine responses to initial conditions and to unit step inputs in rudder and aileron with zero initial conditions. In all cases, the plant followed the candidate model perfectly, as expected. The only remaining question is whether the candidate model behaves dynamically the same as the desired model.

Selected time histories of the responses of the plant (identical to those of the candidate model) and of the desired model are shown in Figs. 1–8. Not shown are the time histories of bank angle and roll rate response to rudder and aileron inputs, in which there was no discernible difference between plant and desired model responses.

Figures 1–4 are the responses due to initial conditions alone, with no control inputs. The initial conditions for this simulation were selected to ensure that all system modes were excited equally. The effects of the small differences between desired and candidate model eigenvectors are noticeable in all four time histories, but the responses are virtually identical.

Figures 5 and 6 show the sideslip and yaw rate response of the plant and the desired model following a unit step rudder input at time zero. Figures 7 and 8 are the same, but for a unit step aileron input. In both cases, the differences in response to control inputs are most pronounced in the sideslip angle time histories. They are, however, dynamically similar in that the describing features of the excited modes (such as frequency and damping of the Dutch roll) have been preserved.

Finally, the error-correcting attributes of the control law are demonstrated in Fig. 9. The initial conditions of the candidate model were taken as $[0 \quad 0 \quad 0 \quad 0]^T$ and of the plant as

$$TA_cT^{-1} = \begin{bmatrix} 3.909 & 0 & 0 & -3.909 \\ 0.383 & -7.938\times10^{-2} & 2.543 & -0.297 \\ -2.229\times10^{-2} & -1.236 & -0.278 & 2.097\times10^{-2} \\ 5.208 & 18.91 & 5.285 & -5.153 \end{bmatrix} \tag{43a}$$

$$TB_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.987 & 2.59\times10^{-4} \\ -1.036\times10^{-2} & 1.001 \end{bmatrix} \tag{43b}$$

Using the control law given by Eq. (22), the error feedback gains were obtained by treating Eq. (23) as a linear quadratic regulator problem with unity weighting. As a result, the error dynamics are

$$\dot{e} = \begin{bmatrix} 3.909 & 0 & 0 & -3.909 \\ 0.383 & -7.938\times10^{-2} & 2.543 & -0.297 \\ -6.70\times10^{-2} & -0.925 & -2.388 & 5.704\times10^{-2} \\ 10.56 & 4.753\times10^{-2} & 5.704\times10^{-2} & -9.141 \end{bmatrix} e \tag{44}$$

Here, the errors are between $y_c$ and $y_p$. This feedback selection results in error eigenvalues of $-1.59 \pm j1.23$ and $-34.7 \pm j33.4$. Finally, the control law is given by

$$u_p = \dot{y}_c^2 + \begin{bmatrix} 6.70\times10^{-2} & 0.925 & 2.388 & -5.704\times10^{-2} \\ -10.56 & -4.753\times10^{-2} & -5.704\times10^{-2} & 9.141 \end{bmatrix} (y_c - y_p)$$

$$- \begin{bmatrix} 0.369 & -0.875 & -0.482 & -0.369 \\ 6.777 & 1.215 & -0.462 & -6.777 \end{bmatrix} y_p \tag{45}$$

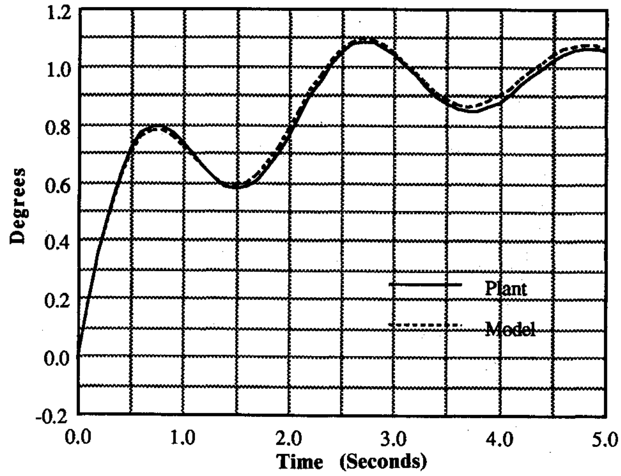


**Fig. 1 Bank angle response to initial conditions.**

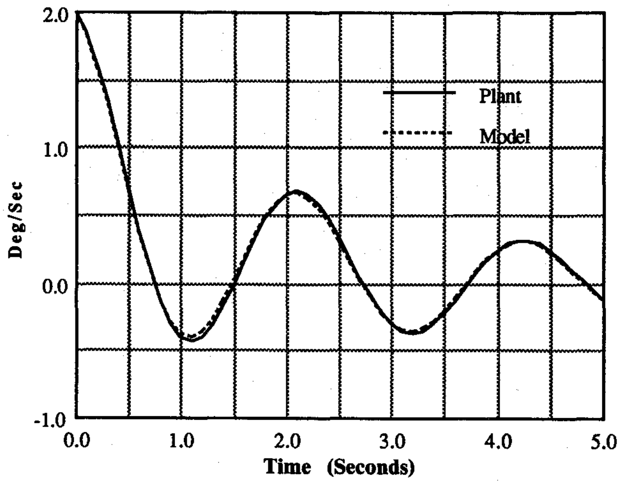


**Fig. 2 Roll rate response to initial conditions.**

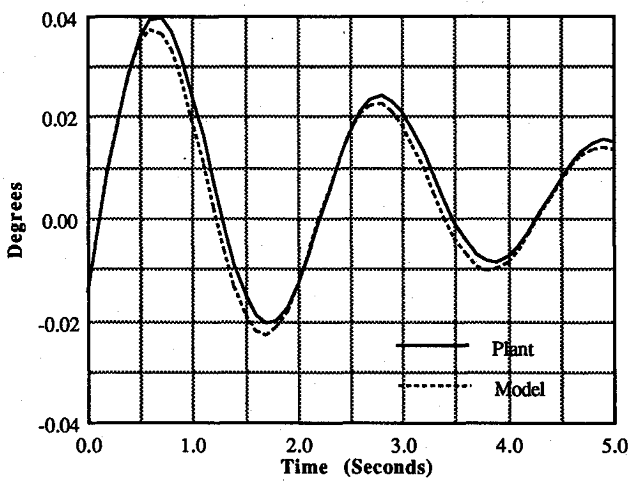


**Fig. 3 Sideslip angle response to initial conditions.**

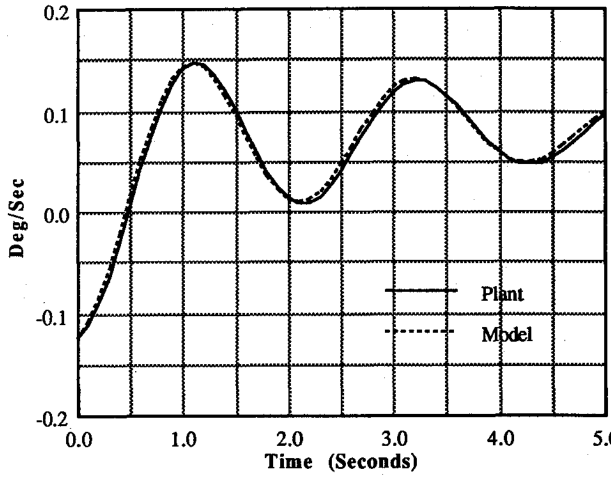


**Fig. 4 Yaw rate response to initial conditions.**

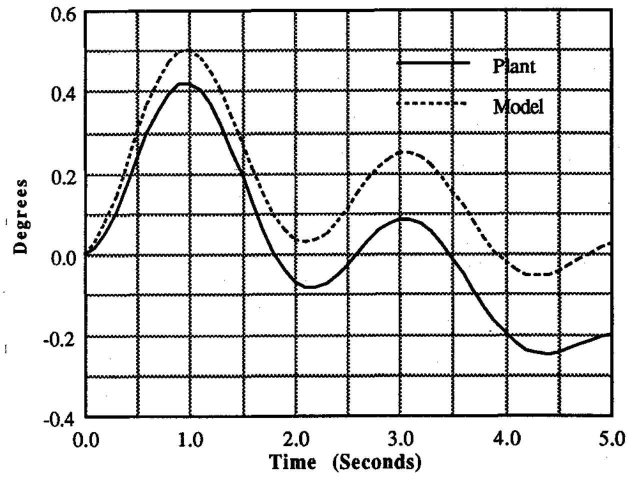


**Fig. 5 Sideslip angle response to rudder input.**

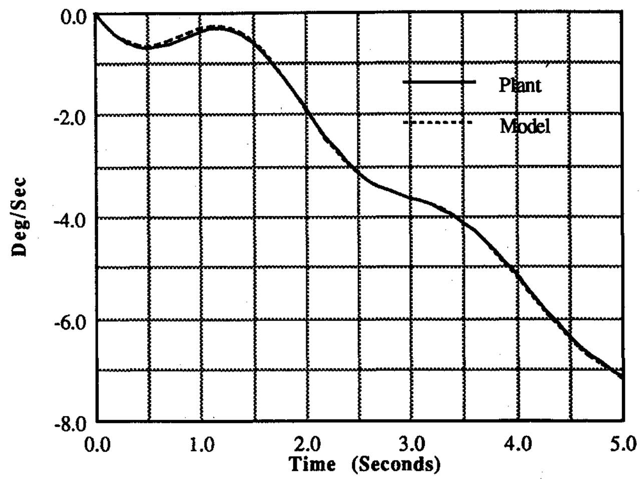


**Fig. 6 Yaw rate response to rudder input.**

$[1 \quad 1 \quad 1 \quad 1]^T$. The systems were excited with step rudder and aileron inputs at time zero. Only the differences between plant and candidate model states are shown in this figure. The initial errors are eliminated effectively within 3 s.

In the interpretation of these results, it is emphasized that it was not the intent of the analysis to make the plant follow the desired model trajectory exactly. The fact that it very nearly does results from the fact that a candidate model was found whose eigenstructure was very nearly the same as that desired. The primary conclusions to be drawn from these time histories are that 1) the control law corrects for errors in initial conditions and causes the plant to follow the candidate model perfectly, and 2) the candidate model has dynamic responses that are similar to those of the desired model.

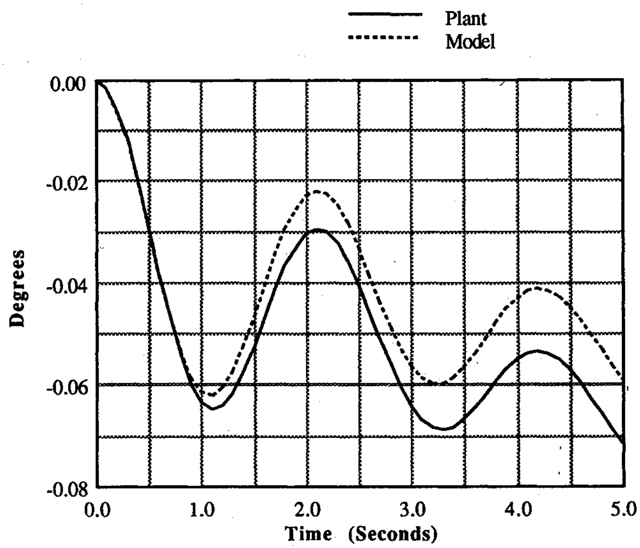


**Fig. 7 Sideslip angle response to aileron input.**

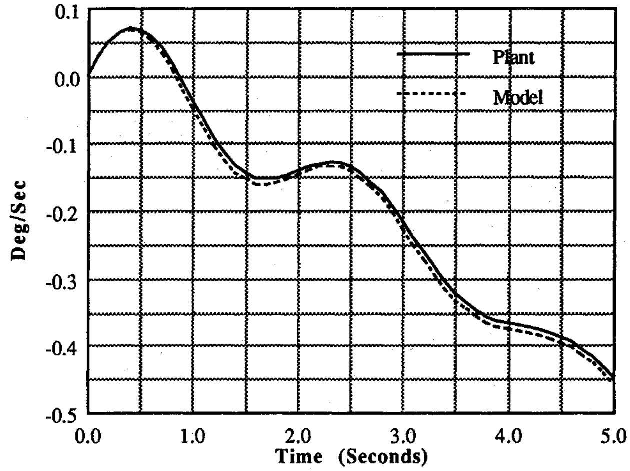


**Fig. 8 Yaw rate response to aileron input.**

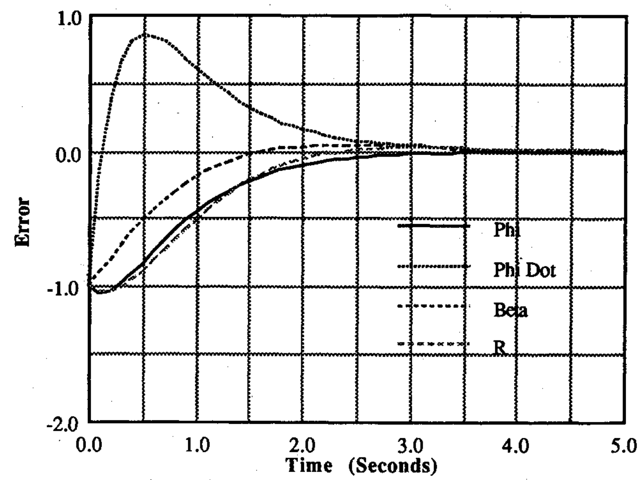


**Fig. 9 Errors, candidate model—plant.**

## Conclusions

Four ideas in model-following control have been presented. The first is the introduction of a standard form for perfect model-following, shown to be a generalization of all pairs of plants and models for which perfect model-following is achievable. The second is a test for perfect model-following that does not presuppose a particular form of the left inverse for the plant control matrix. The third is a formulation of the perfect model-following control law based on the standard form that gives insight into the structure of the system and that allows the arbitrary choice of error dynamics by conventional pole placement methods.

The fourth is an alternative method to solving the imperfect model-following control problem. This approach gives perfect model-following solutions using candidate models whose eigenstructure may be compared directly with that of the desired model. Because the candidate model may be followed perfectly, this means that the controlled plant's eigenstructure may be compared directly with that of the desired model. The degree of imperfection in the solution is measured by differences in these eigenstructures where, for example, it will immediately be obvious how the controlled plant's mode shapes will differ from those desired. This is to be contrasted with the uncertainty in solutions that imperfectly follow a perfect model. In those solutions, the degree of imperfection is not known a priori and is seen only through simulations that are strongly dependent on the selection of initial conditions and control inputs.

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